V Solving ODEs using Laplace Transforms

1 The Laplace Transform

The *Laplace transform* is an integral transform that is useful in solving differential equations.

Definition. Given a function $y(t)$, the *Laplace transform* of y defines a new function Y of a new variable s via the integral relation:

$$Y(s) = \mathcal{L}\{\, y(t) \,\} \equiv \int_{t=0}^{\infty} y(t)e^{-st}\, dt$$

The notation $\mathcal{L}\{\,\cdot\,\}$ is employed to denote taking the Laplace transform of its argument.

Essentially, the Laplace transform "integrates out" the variable t leaving a function of s. However, we are not guaranteed the Laplace transform of a function exists and must ensure we understand the values of s for which it does.

(a) Basic Laplace Transforms

We consider taking the Laplace transform of three simple mathematical functions.

Example. Let $y(t) = k$ where k is some constant, then

$$\mathcal{L}\{\, k \,\} = \int_{t=0}^{\infty} ke^{-st}\, dt = -\frac{ke^{-st}}{s}\bigg|_{t=0}^{\infty} = \lim_{M\to\infty}\left(-\frac{ke^{-sM}}{s}\right) + \frac{k}{s}.$$

The exponential function will *diverge*[1] if $s < 0$ and the integral is not even defined when $s = 0$. However, with $s > 0$ the exponential function e^{-sM} *converges*[2] to zero as $M \to \infty$, and hence

$$\mathcal{L}\{\, k \,\} = \frac{k}{s} \qquad (s > 0).$$

Example. Let $y(t) = e^{-kt}$ where k is some constant, then

$$\begin{aligned}
\mathcal{L}\{\, e^{-kt} \,\} &= \int_{t=0}^{\infty} e^{-kt} e^{-st}\, dt = \int_{t=0}^{\infty} e^{-(s+k)t}\, dt = -\frac{e^{-(s+k)t}}{s+k}\bigg|_{t=0}^{\infty} \\
&= \lim_{M\to\infty}\left(-\frac{e^{-(s+k)M}}{s+k}\right) + \frac{1}{s+k}.
\end{aligned}$$

The exponential function *diverges* if $s + k < 0$ and the result is not defined when $s + k = 0$. For $s + k > 0$ (i.e. $s > -k$), $e^{-(s+k)M} \to 0$ as $M \to \infty$, yielding

$$\mathcal{L}\{\, e^{-kt} \,\} = \frac{1}{s+k} \qquad (s > -k).$$

Example. Let $y(t) = t$ then using integration by parts:

$$\begin{aligned}
\mathcal{L}\{\, t \,\} &= \int_{t=0}^{\infty} te^{-st}\, dt = -\frac{te^{-st}}{s}\bigg|_{t=0}^{\infty} + \frac{1}{s}\int_{t=0}^{\infty} e^{-st}\, dt \\
&= \left[\lim_{M\to\infty}\left(-\frac{Me^{-sM}}{s}\right) - 0\right] + \frac{1}{s} \cdot -\frac{e^{-st}}{s}\bigg|_{t=0}^{\infty} \\
&= \lim_{M\to\infty}\left(-\frac{Me^{-sM}}{s}\right) + \frac{1}{s}\left[\lim_{M\to\infty}\left(-\frac{e^{-sM}}{s}\right) + \frac{1}{s}\right].
\end{aligned}$$

The exponential function *diverges* if $s < 0$ and the result is not defined when $s = 0$. With $s > 0$ we have both $Me^{-sM} \to 0$ and $e^{-sM} \to 0$ as $M \to \infty$, yielding

$$\mathcal{L}\{\, t \,\} = \frac{1}{s^2} \qquad (s > 0).$$

We can continue in this manner and find the Laplace transform for many other mathematical functions. Not much is gained from such calculations so instead we offer the following table:

[1] If $s < 0$ then $e^{-sM} \to \infty$ as $M \to \infty$. A function tending to infinity is said to *diverge*.

[2] Loosely speaking, as M gets larger and larger, e^{sM} gets smaller and smaller.

$y(t)$	$Y(s) \equiv \mathcal{L}\{\, y(t) \,\}$	Condition on s
k	$\dfrac{k}{s}$	$s > 0$
t^n	$\dfrac{n!}{s^{n+1}} = \dfrac{n(n-1)\cdots 3 \cdot 2 \cdot 1}{s^{n+1}}$	$s > 0$
e^{-kt}	$\dfrac{1}{s+k}$	$s > -k$
$\sin(kt)$	$\dfrac{k}{s^2+k^2}$	$s > 0$
$\cos(kt)$	$\dfrac{s}{s^2+k^2}$	$s > 0$

In the above, k denotes a real constant and n denotes *any* positive integer (i.e. $n = 1, 2, 3, 4, \ldots$).

(b) Properties of the Laplace Transform

Due to its definition in terms of an integral, the Laplace transform inherits some properties of integration:

Rule. The Laplace transform is *linear*:

$$\begin{aligned}\mathcal{L}\{\, kf(t)\,\} &= k\mathcal{L}\{\, f(t)\,\} \\ \mathcal{L}\{\, f(t)+g(t)\,\} &= \mathcal{L}\{\, f(t)\,\}+\mathcal{L}\{\, g(t)\,\}\end{aligned}$$

for any functions f, g and constant k.

This property enable us to deal with any *linear combination* of functions in the previous table and will be useful in applications to differential equations. There are two other useful properties we introduce:

Example. Let $y(t) = e^{at}f(t)$ in terms of some constant a and function f. We assume that $\mathcal{L}\{\, f(t)\,\} = F(s)$ for $s > k$ where k is some constant. The Laplace transform of y is:

$$\begin{aligned}\mathcal{L}\{\, e^{at}f(t)\,\} &= \int_{t=0}^{\infty} e^{at}f(t)e^{-st}\,dt = \int_{t=0}^{\infty} f(t)e^{-(s-a)t}\,dt \\ &= F(s-a)\end{aligned}$$

where the final integral exists for $s - a > k$ or $s > k + a$.

Since f was an arbitrary function, this result holds in general:

Rule. Let $\mathcal{L}\{\, f(t)\,\} = F(s)$ for $s > k$ where k is some constant, then

$$\mathcal{L}\{\, e^{at}f(t)\,\} = F(s-a)$$

in terms of some constant a. This is known as the *first shift theorem*.

That is, multiplying by e^{at} 'shifts' the argument of the Laplace transform by a factor of a from s to $s - a$. The first shift theorem is useful in obtaining new Laplace transforms. For example, using our table of Laplace transforms:

$$\begin{aligned}\mathcal{L}\{\, \sin(kt)\,\} = \frac{k}{s^2+k^2} \quad &\Longrightarrow \quad \mathcal{L}\{\, e^{at}\sin(kt)\,\} = \frac{k}{(s-a)^2+k^2} \\ \mathcal{L}\{\, t^2\,\} = \frac{2}{s^3} \quad &\Longrightarrow \quad \mathcal{L}\{\, e^{at}t^2\,\} = \frac{2}{(s-a)^2}.\end{aligned}$$

The following example provides another useful property of Laplace transforms:

Example. Let $y(t) = t \cdot f(t)$ for some function f satisfying $\mathcal{L}\{\, f(t)\,\} = F(s)$ for $s > k$ in terms of some constant k. Then:

$$\begin{aligned}\mathcal{L}\{\, t \cdot f(t)\,\} &= \int_{t=0}^{\infty} t \cdot f(t)e^{-st}\,dt = \int_{t=0}^{\infty} f(t)\left(te^{-st}\right)\,dt \\ &= -\int_{t=0}^{\infty} f(t)\frac{d}{ds}\left(e^{-st}\right)\,dt = -\frac{d}{ds}\int_{t=0}^{\infty} f(t)e^{-st}\,dt \\ &= -\frac{d}{ds}F(s)\end{aligned}$$

where the final integral exists for $s > k$.

Once again, the function f was arbitrary yielding:

Rule. Let $\mathcal{L}\{\, f(t)\,\} = F(s)$ for $s > k$ where k is some constant, then

$$\mathcal{L}\{\, t \cdot f(t)\,\} \quad = \quad -\frac{d}{ds}\,F(s)$$

(c) The Laplace Transform of Derivatives

In order to solve differential equations, we shall need the Laplace transform of derivatives of y. We denote the derivatives using the 'dot' notation so that:

$$\dot{y} = \frac{dy}{dt}, \quad \ddot{y} = \frac{d^2y}{dt^2}, \quad \dddot{y} = \frac{d^3y}{dt^3}, \quad \text{etc.}$$

We work directly from the definition of the Laplace transform:

$$\begin{aligned}
\mathcal{L}\{\, \dot{y}(t)\,\} &= \int_{t=0}^{\infty} \dot{y}(t)e^{-st}\,dt \;=\; y(t)e^{-st}\Big|_{t=0}^{\infty} + s\int_{t=0}^{\infty} y(t)e^{-st}\,dt \\
&= \lim_{M\to\infty}\Big(\, y(t)e^{-sM}\Big) - y(0) + s\mathcal{L}\{\, y(t)\,\},
\end{aligned}$$

by using integration by parts. *If* the first term above vanishes[3], then we have

Rule. For any differentiable function $y = y(t)$:

$$\mathcal{L}\{\, \dot{y}(t)\,\} \quad = \quad s\mathcal{L}\{\, y(t)\,\} - y(0).$$

We will also require the Laplace transform of the second derivative; the procedure follows as above:

$$\begin{aligned}
\mathcal{L}\{\, \ddot{y}(t)\,\} &= \int_{t=0}^{\infty} \ddot{y}(t)e^{-st}\,dt \;=\; \dot{y}(t)e^{-st}\big|_{t=0}^{\infty} + s\int_{t=0}^{\infty} \dot{y}(t)e^{-st}\,dt \\
&= \lim_{M\to\infty}\Big(\, \dot{y}(t)e^{-sM}\Big) - \dot{y}(0) + s\mathcal{L}\{\, \dot{y}(t)\,\} \\
&= \lim_{M\to\infty}\Big(\, \dot{y}(t)e^{-sM}\Big) - \dot{y}(0) + s\Big(\, s\mathcal{L}\{\, y(t)\,\} - y(0)\Big),
\end{aligned}$$

using integration by parts once more. If the first term vanishes[4], then we have

Rule. For any twice-differentiable function $y = y(t)$:

$$\mathcal{L}\{\, \ddot{y}(t)\,\} \quad = \quad s^2\mathcal{L}\{\, y(t)\,\} - sy(0) - \dot{y}(0).$$

The Laplace transform of higher derivatives can also be found, but we shall not require them in this course. When solving differential equations, it is often the case that one *assumes* that the solutions are of exponential order so that these Laplace transforms can be used.

[3] This is true for most functions we are interested in. Specifically, it can be shown to be true for *any* function of *exponential order*.
[4] That is, the derivative of $y(t)$ is also of exponential order.

2 The Inverse Laplace Transforms

We have discussed how to transform $y(t)$ to a new function $Y(s)$ using the Laplace transform. We shall now consider the reverse process: taking a function $Y(s)$ and converting it to a function $y(t)$.

Definition. The *inverse Laplace transform* is denoted $\mathcal{L}^{-1}\{\,\cdot\,\}$ and converts the function Y of the variable s to the function y of the variable t. If $\mathcal{L}\{\,y(t)\,\} = Y(s)$ for some function y, the defining property of the inverse Laplace transform is given by:

$$\mathcal{L}^{-1}\{\,Y(s)\,\} = \mathcal{L}^{-1}\{\,\mathcal{L}\{\,y(t)\,\}\,\} = y(t).$$

As with the Laplace transform, the inverse Laplace transform can be calculated via an integral[5] or we can simply use the table and 'go the other way' since, from the definition above, the inverse Laplace transform 'undoes' taking the Laplace transform of a function. For example, using the table of Laplace transforms:

$$\mathcal{L}\{\,e^{-kt}\,\} = \frac{1}{s+k} \quad\Longrightarrow\quad \mathcal{L}^{-1}\left\{\frac{1}{s+k}\right\} = e^{-kt}.$$

(a) Properties of the Inverse Laplace Transforms

Unfortunately, it is often the case that the functions we wish to take the inverse Laplace transform of are not in a form easily recognisable and cannot be matched within a function in the table of Laplace transforms. Consequently, we must do some further work to put them into a form for which we can use the table. We use the following property in particular:

Definition. The inverse Laplace transform is *linear*:

$$\begin{aligned}\mathcal{L}^{-1}\{\,kF(s)\,\} &= k\mathcal{L}^{-1}\{\,F(s)\,\}\\ \mathcal{L}^{-1}\{\,F(s)+G(s)\,\} &= \mathcal{L}^{-1}\{\,F(s)\,\}+\mathcal{L}^{-1}\{\,G(s)\,\}\end{aligned}$$

for any constant k and Laplace transforms $F(s), G(s)$.

Surprisingly, this property alone is usually enough for us to be able to manipulate expressions into a recognisable form in order to use the table of Laplace transforms.

(b) Calculating Inverse Laplace Transforms

We examine the inverse Laplace transform by considering a number of examples:

Example. Let Y be the function defined by

$$Y(s) = \frac{3}{s-2} + \frac{1}{2s+3} - \frac{5}{s^4},$$

then $\mathcal{L}^{-1}\{\,Y(s)\,\}$ can be found, due to linearity, by considering the inverse Laplace transform of each term. Using

[5] A much harder, more horrible one known as the *Bromwich integral* which is an integral in the complex plane.

the table of Laplace transforms:

$$\frac{3}{s-2} = 3\cdot\frac{1}{s+(-2)} \implies \mathcal{L}^{-1}\left\{\frac{3}{s+2}\right\} = 3e^{-(-2)t} = 3e^{2t}$$

$$\frac{1}{2s+3} = \frac{1}{2(s+\frac{3}{2})} = \frac{1}{2}\cdot\frac{1}{s+\frac{3}{2}} \implies \mathcal{L}^{-1}\left\{\frac{1}{2s+3}\right\} = \frac{1}{2}e^{-\frac{3}{2}t}$$

$$\frac{5}{s^4} = \frac{5}{6}\cdot\frac{6}{s^4} \implies \mathcal{L}^{-1}\left\{\frac{5}{s^4}\right\} = \frac{5}{6}t^3.$$

Using these results we obtain

$$\mathcal{L}^{-1}\{\,Y(s)\,\} = 3e^{2t} + \frac{1}{2}e^{-\frac{3}{2}t} - \frac{5}{6}t^3.$$

Example. Let $Y(s) = \dfrac{4s+7}{3s^2+12} + \dfrac{10+s}{s^2+8s+25}$ and consider each term separately:

$$\begin{aligned}\frac{4s+7}{3s^2+12} &= \frac{4s}{3(s^2+4)} + \frac{7}{3(s^2+4)} = \frac{4}{3}\frac{s}{s^2+2^2} + \frac{7}{3}\frac{1}{s^2+2^2}\\ &= \frac{4}{3}\cdot\frac{s}{s^2+2^2} + \frac{7}{6}\cdot\frac{2}{s^2+2^2}\end{aligned}$$

and hence from the table of Laplace transforms:

$$\mathcal{L}^{-1}\left\{\frac{4s+7}{3s^2+12}\right\} = \frac{4}{3}\cos(2t) + \frac{7}{6}\sin(2t).$$

The second term can be simplified by completing the square in the denominator:

$$\frac{10+s}{s^2+8s+25} = \frac{10+s}{(s+4)^2+9} = \frac{10+s}{(s+4)^2+3^2}.$$

The denominator looks like a shifted version of the Laplace transform of a sine or cosine function; we must make sure the numerator is suitably shifted:

$$\begin{aligned}\frac{10+s}{s^2+8s+25} &= \frac{6+(s+4)}{(s+4)^2+3^2} = \frac{6}{(s+4)^2+3^2} + \frac{s+4}{(s+4)^2+3^2}\\ &= 2\cdot\frac{3}{(s+4)^2+3^2} + \frac{s+4}{(s+4)^2+3^2}.\end{aligned}$$

The table of Laplace transforms (along with the shift theorem) yields:

$$\mathcal{L}^{-1}\left\{\frac{10+s}{s^2+8s+25}\right\} = 2e^{-4t}\sin(3t) + e^{-4t}\cos(3t).$$

Using these results we obtain

$$\mathcal{L}^{-1}\{\,Y(s)\,\} = \frac{4}{3}\cos(2t) + \frac{7}{6}\sin(2t) + 2e^{-4t}\sin(3t) + e^{-4t}\cos(3t).$$

Example. Let $Y(s) = \dfrac{2s}{s^2+s-6}$. This looks similar to the previous example, but completing the square in the denominator yields $(s+\frac{1}{2})^2 - \frac{25}{4}$ which is not of a form recognisable in our table. In this case however, the denominator can be factored:

$$\frac{2s}{s^2+s-6} = \frac{2s}{(s+3)(s-2)} = \frac{A}{s+3} + \frac{B}{s-2}$$

where we now invoke a *partial fraction decomposition*. Multiplying by the factor $(s+3)(s-2)$ yields

$$2s = A(s-2) + B(s+3)$$

which must hold for all s. In particular, with $s = 2$ we find $4 = 5B$ implying $B = \frac{4}{5}$; and for $s = -3$ we find $-6 = -5A$ and hence $A = \frac{6}{5}$. Thus

$$\frac{2s}{s^2+s-6} = \frac{6}{5} \cdot \frac{1}{s+3} + \frac{4}{5} \cdot \frac{B}{s+(-2)}$$

and the table of Laplace transforms gives

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{6}{5}e^{-3t} + \frac{4}{5}e^{2t}.$$

Example. Let $Y(s) = \dfrac{2s-1}{(s-1)(s^2+4)}$. As before, we must use a partial fraction decomposition:

$$\frac{2s-1}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4}.$$

Multiplying both sides by $(s-1)(s^2+4)$ yields

$$2s-1 = A(s^2+4) + (Bs+C)(s-1)$$

which must hold for all values of s. In particular:

$$\begin{aligned}
s=1: &\quad 1 = 5A \implies A = \frac{1}{5} \\
s=0: &\quad -1 = 4A - C \implies C = \frac{9}{5} \\
s=2: &\quad 3 = 8A + 2B + C \implies B = -\frac{1}{5}.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{2s-1}{(s-1)(s^2+4)} &= \frac{1}{5} \cdot \frac{1}{s-1} - \frac{1}{5} \cdot \frac{s}{s^2+4} + \frac{9}{5} \cdot \frac{1}{s^2+4} \\
&= \frac{1}{5} \cdot \frac{1}{s+(-1)} - \frac{1}{5} \cdot \frac{s}{s^2+2^2} + \frac{9}{10} \cdot \frac{2}{s^2+2^2}.
\end{aligned}$$

The table of Laplace transforms yields

$$\mathcal{L}^{-1}\left\{\frac{2s-1}{(s-1)(s^2+4)}\right\} = \frac{1}{5}e^{t} - \frac{1}{5}\cos(2t) + \frac{9}{10}\sin(2t).$$

Using these results we obtain

$$\mathcal{L}^{-1}\{\, Y(s)\,\} \quad = \quad \frac{4}{3}\cos(2t) + \frac{7}{6}\sin(2t) + 2e^{-4t}\sin(3t) + e^{-4t}\cos(3t).$$

3 Linear Inhomogeneous First-order ODEs

We have previously discussed solving linear homogeneous second-order ODEs with constant coefficients but have not attempted the inhomogeneous case. In these next two sections we rectify this omission, employing the method of *Laplace transforms*. To begin, we introduce the method by analysing a much simpler set of problems: solving linear inhomogeneous first-order ODEs with constant coefficients.

(a) The General Method

Consider the following general linear inhomogeneous first-order initial value problem with constant coefficients for $y \equiv y(t)$:

$$\begin{cases} \dot{y} + Ay &= f(t) \\ y(0) &= y_0 \end{cases}$$

where y_0, A are some constants and f is some (known) function of t. We take the Laplace transform of both sides of the ODE:

$$\begin{aligned} \mathcal{L}\{\,\dot{y} + Ay\,\} = \mathcal{L}\{\,f(t)\,\} \quad &\Longrightarrow \quad \mathcal{L}\{\,\dot{y}\,\} + A\mathcal{L}\{\,y\,\} = \mathcal{L}\{\,f(t)\,\} \\ &\Longrightarrow \quad sY(s) - y(0) + AY(s) = F(s) \\ &\Longrightarrow \quad sY(s) - y_0 + AY(s) = F(s) \end{aligned}$$

where we have used the linearity of the Laplace transform, the table of Laplace transforms, the initial condition associated with the ODE and introduced the notation $Y(s) = \mathcal{L}\{\,y\,\}$, $F(s) = \mathcal{L}\{\,f(t)\,\}$. Rearranging we find

$$(s + A)Y(s) = y_0 + F(s) \quad \Longrightarrow \quad Y(s) = \frac{y_0 + F(s)}{s + A}.$$

Thus, if we are able to take the inverse Laplace transform of the left-hand side we have achieved our goal and solved the IVP:

$$y(t) = \mathcal{L}^{-1}\{\,Y(s)\,\} = \mathcal{L}^{-1}\left\{\,\frac{y_0 + F(s)}{s + A}\,\right\}.$$

In essence, taking the Laplace transform of an ODE allow us to 'solve' the problem *algebraically* for $Y(s)$, which is generally much simpler to do, at the expense of having to take the inverse Laplace transform of this solution, which can be tricky.

The Laplace transform is particularly useful when solving differential equations involving *discontinuous functions*[6] which appear frequently in all aspects of engineering.

(b) Some Examples

Example. Consider the system:

$$\begin{cases} \dot{y} + 4y &= e^{-5t} \\ y(0) &= 1 \end{cases}$$

[6] Discontinuous functions are functions that may 'jump' in value at some point. For example, imagine modelling a system with a switch that turns on and off every second; it value at $t = 0$ is 0 (off) then at $t = 1$ is 1 (on), then at $t = 2$ is 0 (off), and so on. This function 'jumps' from 0 to 1 and is known as the *unit step function*.

Taking the Laplace transform of both sides of the ODE and using the initial condition yields:

$$\begin{aligned}
\mathcal{L}\{\,\dot{y}\,\} + 4\mathcal{L}\{\,y\,\} \;=\; \mathcal{L}\{\,e^{-5t}\,\} \qquad &\Longrightarrow \qquad sY(s) - 1 + 4Y(s) \;=\; \frac{1}{s+5} \\
&\Longrightarrow \qquad (s+4)Y(s) \;=\; \frac{1}{s+5} + 1 \;=\; \frac{s+6}{s+5} \\
&\Longrightarrow \qquad Y(s) \;=\; \frac{s+6}{(s+4)(s+5)}
\end{aligned}$$

using the table of Laplace transforms. In order to take the inverse Laplace transform of this expression we must first apply a partial fraction decomposition:

$$\frac{s+6}{(s+4)(s+5)} \;=\; \frac{A}{s+4} + \frac{B}{s+5} \qquad \Longrightarrow \qquad s+6 \;=\; A(s+5) + B(s+4)$$

which must hold for all values of s. In particular, $s=-4$ yields $A=2$ and $s=-5$ yields $B=-1$. Thus we have

$$\frac{s+6}{(s+4)(s+5)} \;=\; \frac{2}{s+4} - \frac{1}{s+5} \qquad \Longrightarrow \qquad y(t) \;=\; \mathcal{L}^{-1}\{\,Y(s)\,\} \;=\; 2e^{-4t} - e^{-5t}$$

using the table of Laplace transforms once again.

We could already solve this first-order ODE using the method of integrating factor, but the method of Laplace transforms generalises more easily to higher-order ODEs, so it is worthwhile understanding this technique for such examples.

Example. Consider the system:

$$\begin{cases} \dot{y} + 2y &= \; 6t^2 \\ \quad y(0) &= \; 0 \end{cases}$$

Taking the Laplace transform of both sides of the ODE and using the initial condition yields:

$$\begin{aligned}
\mathcal{L}\{\,\dot{y}\,\} + 3\mathcal{L}\{\,y\,\} \;=\; 9\mathcal{L}\{\,t^2\,\} \qquad &\Longrightarrow \qquad sY(s) - 0 + 3Y(s) \;=\; \frac{18}{s^3} \\
&\Longrightarrow \qquad (s+3)Y(s) \;=\; \frac{18}{s^3} \\
&\Longrightarrow \qquad Y(s) \;=\; \frac{18}{s^3(s+3)}
\end{aligned}$$

using the table of Laplace transforms. We apply a partial fraction decomposition to $Y(s)$:

$$\frac{18}{s^3(s+3)} \;=\; \frac{A}{s+3} + \frac{Bs^2 + Cs + D}{s^3} \qquad \Longrightarrow \qquad 18 \;=\; As^3 + (Bs^2 + Cs + D)(s+3)$$

which must hold for all values of s. In particular:

$$\begin{aligned}
s = -3: &\qquad 18 \;=\; -27A \qquad \Longrightarrow \qquad A \;=\; -\frac{2}{3} \\
s = 0: &\qquad 18 \;=\; 3D \qquad \Longrightarrow \qquad D \;=\; 6 \\
s = 1: &\qquad 18 \;=\; A + 4B + 4C + 4D \qquad \Longrightarrow \qquad 12B + 12C \;=\; -16 \\
s = 2: &\qquad 18 \;=\; 8A + 20B + 10C + 5D \qquad \Longrightarrow \qquad 60B + 30C \;=\; -20.
\end{aligned}$$

The final relations form a pair of simultaneous equations that may be solved yielding $B = \frac{2}{3}$ and $C = -2$. Thus we have the partial fraction decomposition:

$$\begin{aligned}\frac{18}{s^3(s+3)} &= -\frac{2}{3}\cdot\frac{1}{s+3}+\frac{2}{3}\cdot\frac{1}{s}-\frac{2}{s^2}+6\cdot\frac{1}{s^3}\\ &= -\frac{2}{3}\cdot\frac{1}{s+3}+\frac{2}{3}\cdot\frac{1}{s}-2\cdot\frac{1}{s^2}+3\cdot\frac{2}{s^3}.\end{aligned}$$

The table of Laplace transforms then yields the solution:

$$y(t) = \mathcal{L}^{-1}\{\,Y(s)\,\} = -\frac{2}{3}e^{-3t}+\frac{2}{3}-2t+3t^2.$$

Example. Consider the system:

$$\begin{cases} 2\dot{y}-4y &= e^{3t}\cos(t)\\ \quad y(0) &= 2\end{cases}$$

Taking the Laplace transform of both sides of the ODE and using the initial condition:

$$\begin{aligned}2\mathcal{L}\{\,\dot{y}\,\}-4\mathcal{L}\{\,y\,\} = \mathcal{L}\{\,e^{3t}\cos(t)\,\} \quad &\Longrightarrow \quad 2\Big(\,sY(s)-2\Big)-4Y(s) = \frac{s-3}{(s-3)^2+1}\\ &\Longrightarrow \quad 2(s-2)Y(s) = \frac{s-3}{s^2-6s+10}+4\\ &\Longrightarrow \quad 2(s-2)Y(s) = \frac{4s^2-23s+37}{s^2-6s+10}\\ &\Longrightarrow \quad Y(s) = \frac{1}{2}\cdot\frac{4s^2-23s+37}{(s-2)(s^2-6s+10)}\end{aligned}$$

using the table of Laplace transforms and the shift theorem. We use partial fraction decomposition on $Y(s)$:

$$\frac{4s^2-23s+37}{(s-2)(s^2-6s+10)} = \frac{A}{s-2}+\frac{Bs+C}{s^2-6s+10}$$

$$\Longrightarrow \quad 4s^2-23s+37 = A(s^2-6s+10)+(Bs+C)(s-2)$$

which must hold for all values of s. In particular:

$$\begin{aligned} s=2: &\qquad 7 = 2A \quad\Longrightarrow\quad A = \frac{7}{2}\\ s=0: &\qquad 37 = 10A-2C \quad\Longrightarrow\quad C = -1\\ s=1: &\qquad 18 = 5A-B-C \quad\Longrightarrow\quad B = \frac{1}{2},\end{aligned}$$

which yields

$$\begin{aligned}
\frac{4s^2-23s+37}{(s-2)(s^2-6s+10)} &= \frac{7}{2}\cdot\frac{1}{s-2}+\frac{1}{2}\left(\frac{s-2}{s^2-6s+10}\right)\\
&= \frac{7}{2}\cdot\frac{1}{s+(-2)}+\frac{1}{2}\left(\frac{s-2}{(s-3)^2+1}\right)\\
&= \frac{7}{2}\cdot\frac{1}{s+(-2)}+\frac{1}{2}\left(\frac{(s-3)+1}{(s-3)^2+1}\right)\\
&= \frac{7}{2}\cdot\frac{1}{s+(-2)}+\frac{1}{2}\cdot\frac{s-3}{(s-3)^2+1}+\frac{1}{2}\cdot\frac{1}{(s-3)^2+1}
\end{aligned}$$

after completing the square. The table of Laplace transforms yields the solution:

$$y(t) = \mathcal{L}^{-1}\{\,Y(s)\,\} = \frac{7}{4}e^{2t}+\frac{1}{4}e^{3t}\cos(t)+\frac{1}{4}e^{3t}\sin(t).$$

Exercises

Solve following homogeneous first-order linear ordinary differential equations using the method of Laplace transforms.

1. $\begin{cases} \dot{y}-8y &= 5e^{3t} \\ y(0) &= 1 \end{cases}$

2. $\begin{cases} \dot{y}-2y &= 13\sin(3t) \\ y(0) &= 2 \end{cases}$

3. $\begin{cases} \dot{y}+y &= te^{-t} \\ y(0) &= 4 \end{cases}$

4 Linear Inhomogeneous Second-order ODEs

In this section we extend the method of Laplace transforms in solving linear inhomogeneous first-order ODEs with constant coefficients to those of second-order. The method is precisely the same but the algebraic manipulations are, in general, more involved.

(a) Some Examples

Example. Consider the sysmem:

$$\begin{cases} \ddot{y} + 5\dot{y} + 6y &= 24e^{3t} \\ y(0) &= 1 \\ \dot{y}(0) &= 1 \end{cases}$$

We take the Laplace transform of the ODE using the table of Laplace transforms:

$$\begin{aligned} s^2Y(s) - sy(0) - \dot{y}(0) + 5\Big(sY(s) - y(0)\Big) + 6Y(s) &= \frac{24}{s-3} \\ \implies \qquad s^2Y(s) - s - 1 + 5sY(s) - 5 + 6Y(s) &= \frac{24}{s-3} \\ \implies \qquad (s^2+5s+6)Y(s) &= \frac{24}{s-3} + s + 6 = \frac{s^2+3s+6}{s-3} \\ \implies \qquad Y(s) &= \frac{s^2+3s+6}{(s-3)(s^2+5s+6)} = \frac{s^2+3s+6}{(s-3)(s+2)(s+3)} \end{aligned}$$

In order to take the inverse Laplace transform of $Y(s)$ we use a partial fraction decomposition:

$$\begin{aligned} \frac{s^2+3s+6}{(s-3)(s+2)(s+3)} &= \frac{A}{s-3} + \frac{B}{s+2} + \frac{C}{s+3} \\ \implies \qquad s^2+3s+6 &= A(s+2)(s+3) + B(s-3)(s+3) + C(s-3)(s+2). \end{aligned}$$

This last expression must hold for all values of s and hence:

$$\begin{aligned} s = 3: \qquad 24 &= 30A \implies A = \frac{4}{5} \\ s = -2: \qquad 4 &= -5B \implies B = -\frac{4}{5} \\ s = -3: \qquad 6 &= 6C \implies C = 1, \end{aligned}$$

yielding the partial fraction decomposition:

$$\frac{s^2+3s+6}{(s-3)(s+2)(s+3)} = \frac{4}{5}\cdot\frac{1}{s-3} - \frac{4}{5}\cdot\frac{1}{s+2} + \frac{1}{s+3}.$$

The table of Laplace transforms yields the solution

$$y(t) = \mathcal{L}^{-1}\{\,Y(s)\,\} = \frac{4}{5}e^{3t} - \frac{4}{5}e^{-2t} + e^{-3t}.$$

Example. Consider the system:

$$\begin{cases} \ddot{y} - 2\dot{y} + 2y &= 10t \\ y(0) &= 0 \\ \dot{y}(0) &= 0 \end{cases}$$

We take the Laplace transform of the ODE using the table of Laplace transforms:

$$\begin{aligned} s^2Y(s) - sy(0) - \dot{y}(0) - 2\Big(sY(s) - y(0)\Big) + 2Y(s) \;&=\; \frac{10}{s^2} \\ \implies \qquad s^2Y(s) - 2sY(s) + 2Y(s) \;&=\; \frac{10}{s^2} \\ \implies \qquad (s^2 - 2s + 2)Y(s) \;&=\; \frac{10}{s^2} \\ \implies \qquad Y(s) \;&=\; \frac{10}{s^2(s^2-2s+2)}. \end{aligned}$$

In order to take the inverse Laplace transform of $Y(s)$ we use a partial fraction decomposition:

$$\begin{aligned} \frac{10}{s^2(s^2-2s+2)} \;&=\; \frac{As+B}{s^2} + \frac{Cs+D}{s^2-2s+2} \\ \implies \qquad 10 \;&=\; (As+B)(s^2-2s+2) + (Cs+D)s^2. \end{aligned}$$

This last expression must hold for all values of s and hence:

$$\begin{aligned} s=0: &\qquad 10 \;=\; 2B \quad\implies\quad B \;=\; 5 \\ s=-1: &\qquad 10 \;=\; -5A+5B-C+D \quad\implies\quad 5A+C-D \;=\; 15 \\ s=1: &\qquad 10 \;=\; A+B+C+D \quad\implies\quad A+C+D \;=\; 5 \\ s=2: &\qquad 10 \;=\; 4A+2B+8C+4D \quad\implies\quad 4A+8C+4D \;=\; 0. \end{aligned}$$

This system of three equations in three unknowns can be solved; we add the second equation to the third and four times the first to the last:

$$\begin{aligned} 6A+2C &= 20 \\ 24A+12C &= 60 \end{aligned} \quad\implies\quad \begin{aligned} 24A+8C &= 80 \\ 24A+12C &= 60 \end{aligned} \quad\implies\quad 4C \;=\; -20 \quad\implies\quad C \;=\; -5.$$

This in turn yields $A = \frac{1}{6}(20-2C) = 5$ and from our third equation $D = 5 - A - C = 5$. Thus we arrive at the partial fraction decomposition:

$$\frac{10}{s^2(s^2-2s+2)} \;=\; \frac{5s+5}{s^2} - \frac{5s-5}{s^2-2s+2} \;=\; 5\cdot\frac{1}{s} + 5\cdot\frac{1}{s^2} - 5\cdot\frac{s-1}{(s-1)^2+1}.$$

The table of Laplace transforms and the shift theorem yields the solution

$$y(t) \;=\; \mathcal{L}^{-1}\{\,Y(s)\,\} \;=\; 5 + 5t - 5e^t\cos(t).$$

(b) Undamped Forced Oscillations

We return to a previous topic: modelling vehicle suspension using the 1/8-car model. We obtained the ODE[7]

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x \;=\; 2\zeta\omega_0\dot{y} + \omega_0^2 y$$

where $x = x(t)$, $y = y(t)$, ω_0 denotes the natural frequency of the system and ζ is the damping coefficient. Previously, we discussed the case $y(t) = 0$ but we are now in a position to consider the inhomogeneous case $(y(t) \neq 0)$[8]. To simplify matters, we assume there is no damping ($\zeta = 0$) and consider the case $y(t) = d\cos(\omega t)$ where ω, d are constants denoting the frequency and amplitude of this driving term (or of the road profile) respectively. Thus, we have the ODE system:

$$(\star) \quad \begin{cases} \ddot{x} + \omega_0^2 x &= \omega_0^2 d\cos(\omega t) \\ x(0) &= \alpha \\ \dot{x}(0) &= \beta \end{cases}$$

where α, β are some constants. Taking the Laplace transform of the ODE:

$$\begin{aligned} s^2X(s) - sx(0) - \dot{x}(0) + \omega_0^2 X(s) \;&=\; \omega_0^2 d\,\mathcal{L}\{\cos(\omega t)\} \\ \Longrightarrow \qquad s^2X(s) - \alpha s - \beta + \omega_0^2 X(s) \;&=\; \frac{\omega_0^2 ds}{s^2+\omega^2} \\ \Longrightarrow \qquad (s^2+\omega_0^2)X(s) \;&=\; \frac{\omega_0^2 ds}{s^2+\omega^2} + \alpha s + \beta \;=\; \frac{\alpha s^3 + \beta s^2 + (\omega_0^2 d + \alpha\omega^2)s + \beta\omega^2}{s^2+\omega^2} \\ \Longrightarrow \qquad X(s) \;&=\; \frac{\alpha s^3 + \beta s^2 + (\omega_0^2 d + \alpha\omega^2)s + \beta\omega^2}{(s^2+\omega^2)(s^2+\omega_0^2)}. \end{aligned}$$

In order to proceed, we invoke a partial fraction decomposition of $Y(s)$:

$$\begin{aligned} \frac{\alpha s^3 + \beta s^2 + (\omega_0^2 d + \alpha\omega^2)s + \beta\omega^2}{(s^2+\omega^2)(s^2+\omega_0^2)} \;&=\; \frac{As+B}{s^2+\omega_0^2} + \frac{Cs+D}{s^2+\omega^2} \\ \Longrightarrow \quad \alpha s^3 + \beta s^2 + (\omega_0^2 d + \alpha\omega^2)s + \beta\omega^2 \;&=\; (As+B)(s^2+\omega^2) + (Cs+D)(s^2+\omega_0^2). \end{aligned}$$

This final equality must hold for all values of s. In particular, $s = -1$, $s = 0$, $s = 1$ and $s = 2$ yields four equations in the four unknowns A, B, C, D which may be solved to yield:

$$A \;=\; \alpha - \frac{\omega_0^2 d}{\omega_0^2 - \omega^2}, \quad B \;=\; \beta, \quad C \;=\; \frac{\omega_0^2 d}{\omega_0^2 - \omega^2}, \quad D \;=\; 0.$$

Thus we obtain

$$\begin{aligned} X(s) \;&=\; A\cdot\frac{s}{s^2+\omega_0^2} + \frac{B}{\omega_0^2}\cdot\frac{\omega_0^2}{s^2+\omega_0^2} + C\cdot\frac{s}{s^2+\omega^2} + \frac{D}{\omega^2}\cdot\frac{\omega^2}{s^2+\omega^2} \\ &=\; \left(\alpha - \frac{\omega_0^2 d}{\omega_0^2-\omega^2}\right)\cdot\frac{s}{s^2+\omega_0^2} + \frac{\beta}{\omega_0}\cdot\frac{\omega_0}{s^2+\omega_0^2} + \left(\frac{\omega_0^2 d}{\omega_0^2-\omega^2}\right)\cdot\frac{s}{s^2+\omega^2}. \end{aligned}$$

[7]Note we have made a slight change ($\omega \to \omega_0$), adding a subscript to ω to indicate that it is the *natural frequency* of the system under consideration.

[8]Recall that the function $y(t)$ can be considered to determine the profile of the road over which the vehicle is travelling or equally as describing any background vibrations.

The table of Laplace transforms yields the solution:

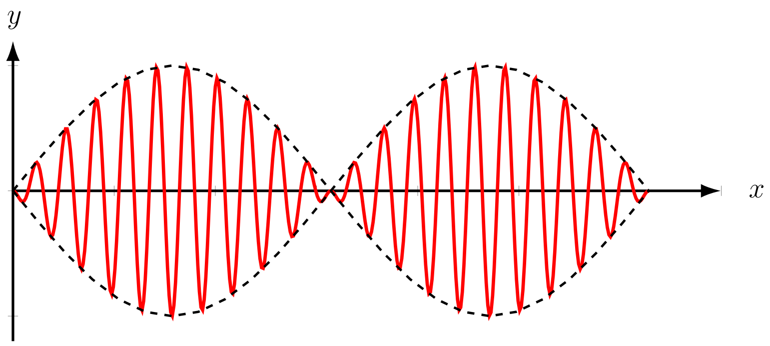
$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \left(\alpha - \frac{\omega_0^2 d}{\omega_0^2-\omega^2}\right)\cos(\omega_0 t) + \frac{\beta}{\omega_0}\sin(\omega_0 t) + \left(\frac{\omega_0^2 d}{\omega_0^2-\omega^2}\right)\cos(\omega t)$$

$$= \alpha\cos(\omega_0 t) + \frac{\beta}{\omega_0}\sin(\omega_0 t) + \left(\frac{\omega_0^2 d}{\omega_0^2-\omega^2}\right)\Big(\cos(\omega t) - \cos(\omega_0 t)\Big).$$

This last expression can be combined using a sum-to-product trigonometric relation to yield

$$x(t) = \alpha\cos(\omega_0 t) + \frac{\beta}{\omega_0}\sin(\omega_0 t) + \left(\frac{2\omega_0^2 d}{\omega_0^2-\omega^2}\right)\sin\left(\frac{1}{2}(\omega_0+\omega)t\right)\sin\left(\frac{1}{2}(\omega_0-\omega)t\right).$$

We recognise the first two terms as the solutions to the homogeneous problem; the final term is due to the inhomogeneity arising from the forcing (or road profile). Notice that as $\omega \to \omega_0$ then the amplitude of this final term tends to infinity. This excitation of large oscillations by the matching of the natural frequency of the system and the frequency of the input is known as *resonance*. Another important type of oscillation is obtained when ω is close to ω_0: *beats*. For such a case $\omega_0 - \omega$ is small and the period of the last sine wave in our inhomogeneous solution is large. The resulting oscillation effectively has its amplitude modified by the lower frequency sine wave component and we obtain an oscillation of the type below:



It is an effect that is commonly used by musicians to check the tuning of their stringed instruments and occurs due to a periodic cancellation of the vibrations of the strings at a slow frequency. The same process is used in automotive radar technologies, both for advanced safety and parking systems and also by law enforcement for speed detection.

(c) Resonance

For the case of resonance ($\omega = \omega_0$) we must return to the ODE system ($\star$). For simplicity, we consider

$$(\star\star) \quad \begin{cases} \ddot{x} + \omega_0^2 x &= \omega_0^2 d\cos(\omega_0 t) \\ x(0) &= 0 \\ \dot{x}(0) &= 0 \end{cases}$$

so that the system starts at $x = 0$ from rest $\dot{x} = 0$. The Laplace transform of the ODE yields

$$s^2 X(s) - sx(0) - \dot{x}(0) + \omega_0^2 X(s) = \omega_0^2 d\,\mathcal{L}\{\cos(\omega_0 t)\}$$

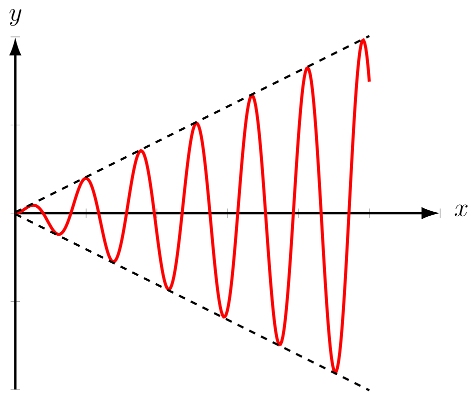
$$\Longrightarrow \quad (s^2+\omega_0^2)X(s) = \frac{\omega_0^2 s d}{s^2+\omega_0^2}$$

$$\Longrightarrow \quad X(s) = \frac{\omega_0^2 s d}{(s^2+\omega_0^2)^2} = -\frac{1}{2}\omega_0 d\cdot\frac{d}{ds}\left(\frac{\omega_0}{s^2+\omega_0^2}\right) = -\frac{1}{2}\omega_0 d\cdot\frac{d}{ds}\,\mathcal{L}\{\sin(\omega_0 t)\}.$$

Thus, the table of Laplace transforms yields the solution

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{2}\omega_0 d\,t\sin(\omega_0 t).$$

Due to the factor t in this solution, the amplitude of the oscillations grows larger and larger:



Practically, systems with very little damping (or none in this case) may undergo large vibrations that can effectively destroy the system. In this sense, damping is a positive feature for systems displaying mechanical vibrations.

Exercises

Solve the following homogeneous first-order linear ordinary differential equations using the method of Laplace transforms.

1. $$\begin{cases} \ddot{y} - 2\dot{y} + y &= 2t - 3 \\ y(0) &= 5 \\ \dot{y}(0) &= 11 \end{cases}$$

2. $$\begin{cases} \ddot{y} + 8\dot{y} + 25y &= 13e^{-2t} \\ y(0) &= -1 \\ \dot{y}(0) &= 18 \end{cases}$$

3. $$\begin{cases} \ddot{y} + 4y &= 8\cos(2t) - 8e^{-2t} \\ y(0) &= -2 \\ \dot{y}(0) &= 0 \end{cases}$$